Differences in positions along a hierarchy: Counterfactuals based on an assignment model

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Online Appendix

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In this Online Appendix, we prove that $N\left(u^{\lfloor vN \rfloor} \left| X \right.\right)/N \underset{a.s.}{\rightarrow} n\left(v \left| X \right.\right)$ for all $v \in (0,1)$ where gender j enters the set of observable charateristics to simplify the notations. The proof of consistency can be decomposed into three stages:

- We first consider a setting where $\beta(.)$ is constant and we establishes consistency using statistical results proposed by Rosén (1972).
- We then extend the consistency result to the case where $\beta(.)$ is piece-wise constant. This is done by proving consistency on each interval.
- We finally extend the consistency result to the case where $\beta(.)$ is a continuous function by using a piece-wise constant approximation of it.

1/ Case of a constant function $\beta(.)$

Lemma B1 (Rosén, 1972). Consider the sampling without replacement of N individuals with weights p_i for $i = 1, ..., N$, such that $\sum_{i=1}^{N} p_i = 1$. Define implicitly the function $t(.)$ with the equation:

$$N - y = \sum_{i=1}^{N} \exp\left(-p_i t\left(y\right)\right) \tag{29}$$

Attribute to each individual i a quantity a_i and define $Z_k = \sum_{i=1}^{N} W_i a_i$ where W_i is a dummy for individual i to be in the sample constituted by the first k draws. We then have:

$$\begin{aligned} EZ_k &= \sum_{i=1}^{N} a_i \left(1 - \exp\left[-p_i t\left(k\right)\right]\right) - r_E\left(N-k\right) N^{\frac{1}{2}} && (30) \\ VZ_k &= \sum_{i=1}^{N} \left[a_i - \frac{\sum_{i=1}^{N} p_i a_i \exp\left[-p_i t\left(k\right)\right]}{\sum_{i=1}^{N} p_i \exp\left[-p_i t\left(k\right)\right]}\right]^2 \left[1 - \exp\left[-p_i t\left(k\right)\right]\right] \exp\left[-p_i t\left(k\right)\right] + r_V\left(k\right) && (31) \end{aligned}$$

where $\lim_{N\to\infty} \max_{\tau_1 \le k/N \le \tau_2} \left|r_E\left(k\right)\right| = 0$ and $\lim_{N\to\infty} \max_{\tau_1 \le k/N \le \tau_2} \left|r_V\left(k\right)/N\right| = 0$ for all $0 < \tau_1 < \tau_2 < 1$.

Denote $k = \lfloor uN \rfloor$ with $\lfloor \cdot \rfloor$ denoting the integer part for $u \in (0,1)$ where we omit the dependance of k with respect to u and N for readability in the proofs, and by u^k the k^{th} empirical rank which is given by $u^k = k/N$. We are going to show that:

$$\frac{N\left(u^k \left| X \right.\right)}{N} = n\left(u \left| X \right.\right) + O\left(N^{-1/2}\right) \tag{32}$$

The roadmap of the proof is the following:

1. Using Lemma B1, we will first show that:

$$E\left[N\left(u^k\left|X\right.\right)\right] = N\left(X\right)\exp\left[-p_X^N t_N\left(1-k/N\right)\right] + r_E\left(k\right)N^{\frac{1}{2}} \tag{33}$$

with $\lim_{N\to\infty}\max_{\tau_1 N\leq k\leq \tau_2 N}\left|r_E\left(k\right)\right| = 0$ for every $0<\tau_1<\tau_2<1$, $N\left(X\right)$ the number of individuals with characteristics X in the population and $t_N\left(u\right) = t\left(Nu\right)/N$ with $t\left(\bullet\right)$ defined according to (29) with $p_i = p_{X_i}^N$ where:

$$p_X^N = \frac{\exp\left(X\beta\right)}{E_{X,emp}\left[\exp\left(X\beta\right)\right]} \tag{34}$$

with $E_{X,emp}$ the empirical expectation computed using the empirical distribution of X such that:

$$E_{X,emp}\left[\exp\left(X\beta\right)\right] = \sum_{\ell} N\left(X^{\ell}\right)\exp\left(X^{\ell}\beta\right)/N \tag{35}$$

2. We will establish the intermediary result that $t_N\left(u\right)$ converges almost surely to $t_\infty\left(u\right)$ over $[0,1)$, where t_∞ is implicitely defined by the equation:

$$1-u = E_X\left[\exp\left(-p_X t_\infty\left(u\right)\right)\right] \tag{36}$$

where E_X is the expectation computed using the aymptotic distribution of X and:

$$p_X = \frac{\exp\left(X\beta\right)}{E_X\left[\exp\left(X\beta\right)\right]} \tag{37}$$

and show that we have:

$$N^{1/2}\left(t_N\left(u\right) - t_\infty\left(u\right)\right) \xrightarrow{L} N\left(0, \frac{1}{\left(1-u\right)^2}\overrightarrow{g}\left(u\right)' V \overrightarrow{g}\left(u\right)\right) \tag{38}$$

where V is a matrix which terms (l,l') for $l\neq l'$ are given by $-n\left(X^l\right)n\left(X^{l'}\right)$ and for $l=l'$ are given by $n\left(X^l\right)\left[1-n\left(X^l\right)\right]$, and $\overrightarrow{g}\left(u\right) = \left[g\left(X^1,u\right),...,g\left(X^L,u\right)\right]'$ with $g\left(X,u\right) = p_X t_\infty\left(u\right)E_X\left[p_X\exp\left(-p_X t_\infty\left(u\right)\right)\right] + \exp\left[-p_X t_\infty\left(u\right)\right]$.

3. From 1/ and 2/, we will deduce that:

$$\frac{E\left[N\left(u^k\left|X\right.\right)\right]}{N} = n\left(u\left|X\right.\right) + o\left(N^{-1/2}\right) \tag{39}$$

4. Finally, we will get that:

$$\frac{N\left(u^k\left|X\right.\right)}{N} = n\left(u\left|X\right.\right) + O\left(N^{-1/2}\right) \tag{40}$$

Step 1 can be shown by applying Lemma B1 with $p_i = p_X^N$ being the probability that an individual has characteristics X, $a_i = 1_{i \in X}$ being a dummy for individual i having characteristics X and the sum (30) being computed for $N-k$ draws. We then have $Z_{N-k} = \sum_{i=1}^{N} \left(1 - 1_{i \in \Omega(u^k)}\right) 1_{i \in X}$ where $1_{i \in \Omega(u^k)}$ denotes the dummy for still being in the risk set after $N-k$ draws such that:

$$EZ_{N-k} = N(X)\left[1 - \exp\left[-p_X^N t(N-k)/N\right]\right] - r_E(k)\, N^{\frac{1}{2}} \tag{41}$$

As we have:

$$Z_{N-k} = \sum_{i=1}^{N} \left(1_{i \in X} - 1_{i \in \Omega(u^k|X)}\right) = N(X) - N\left(u^k|X\right) \tag{42}$$

equation (41) can be rewritten as:

$$N(X) - E\left[N\left(u^k|X\right)\right] = N(X)\left[1 - \exp\left[-p_X^N t_N(1-k/N)\right]\right] - r_E(k)\, N^{\frac{1}{2}} \tag{43}$$

with $t_N(u) = t(Nu)/N$. Rearranging the terms, we get expression (33).

Step 2 can be completed by first showing that $t_N(u)$ converges almost surely to $t_\infty(u)$ over $[0,1)$. Rewriting (29) applied to $p_i = p_{X_i}^N$ where X_i are the characteristics of individual i and $y = Nu$, we get:

$$N - Nu = \sum_{i=1}^{N} \exp\left(-p_{X_i}^N t(Nu)\right) \tag{44}$$

Considering the definition of t_N and the fact that the characteristics of individuals are distributed across the values X^ℓ with $\ell = 1, ..., N$, we get:

$$1 - u = E_{X,emp}\left[\exp\left[-p_X^N t_N(u)\right]\right] \tag{45}$$

We have $N\left(X^\ell\right)/N \to n\left(X^\ell\right)$ almost surely and thus:

$$\lim_N p_X^N = \frac{\exp(X\beta)}{\sum_\ell n(X^\ell)\exp(X^\ell\beta)} = \frac{\exp(X\beta)}{E_X\left[\exp(X\beta)\right]} \equiv p_X \tag{46}$$

Since function $f(x) = E_X\left[\exp(-p_X x)\right]$ is continuous and monotonic, we finally have $\lim_{N\to\infty} t_N(u) = t_\infty(u)$ for all $u \in [0,1)$ with:

$$1 - u = E_X\left[\exp\left(-p_X t_\infty(u)\right)\right] \tag{47}$$

We now establish the asymptotic distribution of t_N. For that purpose, we establish the asymptotic distribution of $E_X\left[\exp\left(-p_X t_N(u)\right)\right]$ and apply the delta method to the inverse of the function $f(\bullet)$. We have the decomposition:

$$\begin{aligned} E_X\left[\exp\left(-p_X t_N(u)\right)\right] - E_X\left[\exp\left(-p_X t_\infty(u)\right)\right] &= E_X\left[\exp\left(-p_X t_N(u)\right)\right] - E_X\left[\exp\left(-p_X^N t_N(u)\right)\right] \\ &\quad + E_X\left[\exp\left(-p_X^N t_N(u)\right)\right] - E_X\left[\exp\left(-p_X t_\infty(u)\right)\right] \end{aligned} \tag{48}$$

Rearranging the first right-hand side term in (48), we get:

$$E_X\left[\exp\left(-p_X t_N(u)\right)\right] - E_X\left[\exp\left(-p_X^N t_N(u)\right)\right] = E_X\left[\exp\left(-p_X t_N(u)\right)\left[1 - \exp\left(-\left(p_X^N - p_X\right) t_N(u)\right)\right]\right] \tag{49}$$

To develop further this expression, we first derive useful properties for $p_X^N - p_X$. Using the fact that p_X^N converges to p_X, we have:

$$p_X^N - p_X = \frac{\exp(X\beta)}{E_{X,emp}\left[\exp(X\beta)\right]} - \frac{\exp(X\beta)}{E_X\left[\exp(X\beta)\right]} \tag{50}$$

$$= -\frac{\exp(X\beta)}{E_{X,emp}\left[\exp(X\beta)\right] E_X\left[\exp(X\beta)\right]}\left[E_{X,emp}\left[\exp(X\beta)\right] - E_X\left[\exp(X\beta)\right]\right] \tag{51}$$

$$= -\frac{\exp(X\beta)}{E_{X,emp}\left[\exp(X\beta)\right] E_X\left[\exp(X\beta)\right]}\sum_\ell \left[N\left(X^\ell\right)/N - n\left(X^\ell\right)\right]\exp\left(X^\ell\beta\right) \tag{52}$$

$$= -p_X^N \sum_\ell \left[N\left(X^\ell\right)/N - n\left(X^\ell\right)\right] p_{X^\ell} \tag{53}$$

Using the fact that $p_X^N = \left(p_X^N - p_X\right) + p_X$ on the right-hand side and rearranging terms by passing those involving $p_X^N - p_X$ on the left-hand side, we get:

$$p_X^N - p_X = -p_X \frac{\sum_\ell \left[N\left(X^\ell\right)/N - n\left(X^\ell\right)\right] p_{X^\ell}}{1 + \sum_\ell \left[N\left(X^\ell\right)/N - n\left(X^\ell\right)\right] p_{X^\ell}} \tag{54}$$

Using the central limit theorem, we have $N\left(X^\ell\right)/N - n\left(X^\ell\right) = O\left(N^{-1/2}\right)$. Using a Taylor expansion of the right-hand side denominator of (54), we then get that:

$$p_X^N - p_X = -p_X \sum_\ell \left[N\left(X_\ell\right)/N - n\left(X_\ell\right)\right] p_{X^\ell} + O\left(N^{-1}\right) \tag{55}$$

Injecting (55) into (49) and making a Taylor expansion, we get:

$$\begin{aligned}
& E_X\left[\exp\left(-p_X t_N(u)\right)\right] - E_X\left[\exp\left(-p_X^N t_N(u)\right)\right] \\
= & -E_X\left[\exp\left(-p_X t_N(u)\right)\left[p_X \sum_{\ell}\left[N\left(X^{\ell}\right)/N - n\left(X^{\ell}\right)\right] p_{X^{\ell}} t_N(u) + O\left(N^{-1}\right)\right]\right] \quad (56) \\
= & -\left[\sum_{\ell}\left[N\left(X^{\ell}\right)/N - n\left(X^{\ell}\right)\right] p_{X^{\ell}}\right] E_X\left[p_X t_N(u) \exp\left(-p_X t_N(u)\right)\right] + O\left(N^{-1}\right) \quad (57)
\end{aligned}$$

Using the fact that $t_N(u) \to t_\infty(u)$ almost surely, equation (57) becomes:

$$\begin{aligned}
& E_X\left[\exp\left(-p_X t_N(u)\right)\right] - E_X\left[\exp\left(-p_X^N t_N(u)\right)\right] \\
= & -\left[\sum_{\ell}\left[N\left(X^{\ell}\right)/N - n\left(X^{\ell}\right)\right] p_{X_{\ell}}\right] t_\infty(u)\, E_X\left[p_X \exp\left(-p_X t_\infty(u)\right)\right] + o\left(N^{-1/2}\right) \quad (58)
\end{aligned}$$

The second right-hand side term in (48) can be rewritten considering that by definition:

$$E_X\left[\exp\left(-p_X t_\infty(u)\right)\right] = 1 - u = \sum_{\ell} N\left(X^{\ell}\right) \exp\left[-p_{X^{\ell}}^N t_N(u)\right]/N \quad (59)$$

This yields:

$$\begin{aligned}
& E_X\left[\exp\left(-p_X^N t_N(u)\right)\right] - E_X\left[\exp\left(-p_X t_\infty(u)\right)\right] \\
= & -\sum_{\ell}\left[N\left(X^{\ell}\right)/N - n\left(X^{\ell}\right)\right] \exp\left[-p_{X^{\ell}}^N t_N(u)\right] \quad (60) \\
= & -\sum_{\ell}\left[N\left(X^{\ell}\right)/N - n\left(X^{\ell}\right)\right] \exp\left[-p_{X^{\ell}} t_\infty(u)\right] + o\left(N^{-1/2}\right) \quad (61)
\end{aligned}$$

Substituting (58) and (61) into (48) gives:

$$\begin{aligned}
& E_X\left[\exp\left(-p_X t_N(u)\right) - E_X\left[\exp\left(-p_X t_\infty(u)\right)\right]\right] \\
= & -\sum_{\ell}\left[N\left(X^{\ell}\right)/N - n\left(X^{\ell}\right)\right] g\left(X^{\ell}, u\right) + o\left(N^{-1/2}\right) \quad (62)
\end{aligned}$$

where $g(X,u) = p_X t_\infty(u)\, E_X\left[p_X \exp\left(-p_X t_\infty(u)\right)\right] + \exp\left[-p_X t_\infty(u)\right]$.

The vector $\left[N\left(X^1\right)/N, ..., N\left(X^L\right)/N\right]'$ contains the averages of dummies for the categories of a multinomial law giving to an individual i a value of explanatory variables X_i. Its expectation is $\left[n\left(X^1\right), ..., n\left(X^L\right)\right]'$ and its law is asymptotically normal with an asymptotic covariance matrix V which terms (l, l') for $l \neq l'$ are given by $-n\left(X^l\right) n\left(X^{l'}\right)$ and for $l = l'$ are given by $n\left(X^l\right)\left[1 - n\left(X^l\right)\right]$. Defining $\overrightarrow{g}(u) = \left[g\left(X^1, u\right), ..., g\left(X^L, u\right)\right]'$ and applying the delta method, we obtain:

$$N^{1/2}\left(E_X\left[\exp\left(-p_X t_N\left(u\right)\right)\right]-E_X\left[\exp\left(-p_X t_\infty\left(u\right)\right)\right]\right)\xrightarrow{L}N\left(0,\overrightarrow{g}\left(u\right)'V\overrightarrow{g}\left(u\right)\right) \tag{63}$$

Applying the delta method again to the inverse of function $f\left(x\right)=E_X\left[\exp\left(-p_X x\right)\right]$ which derivative is given by $f^{-1\prime}\left(y\right)=1/f'\left(f^{-1}\left(y\right)\right)=1/E_X\left[\exp\left(-p_X f^{-1}\left(y\right)\right)\right]$, and considering the equality (47), we finally get the expression:

$$N^{1/2}\left(t_N\left(u\right)-t_\infty\left(u\right)\right)\xrightarrow{L}N\left(0,\frac{1}{\left(1-u\right)^2}\overrightarrow{g}\left(u\right)'V\overrightarrow{g}\left(u\right)\right) \tag{64}$$

Step 3 can be shown from (33) using Taylor expansions. Indeed, we have by definition that $|u-k/N|\leq 1/N$. We also have from (55) that $p_X^N-p_X=O(N^{-1/2})$ since $N\left(X\right)/N-n\left(X\right)=O\left(N^{-1/2}\right)$, and from (64) that $t_N\left(1-u\right)=t_\infty\left(1-u\right)+O(N^{-1/2})$. This yields that:

$$\exp\left[-p_X^N t_N\left(1-k/N\right)\right]=\exp\left[-p_X t_\infty\left(1-u\right)\right]+O\left(N^{-1/2}\right) \tag{65}$$

and thus, we get from (33):

$$E\left[N\left(u^k\left|X\right.\right)\right]/N=n\left(X\right)\exp\left[-p_X t_\infty\left(1-u\right)\right]+O\left(N^{-1/2}\right) \tag{66}$$

Denoting $\overline{n}_X\left(u\right)=n\left(X\right)\exp\left(-p_X t_\infty\left(1-u\right)\right)$, it is possible to establish that $\overline{n}_X\left(u\right)=n\left(u\left|X\right.\right)$ which will complete the proof. This can be done by deriving $\overline{n}_X\left(u\right)$, as we have:

$$\overline{n}'_X\left(u\right)=p_X t'_\infty\left(1-u\right)n\left(X\right)\exp\left(-p_X t_\infty\left(1-u\right)\right) \tag{67}$$

and deriving equation (36), we get:

$$t'_\infty\left(1-u\right)E_X\left[p_X\exp\left(-p_X t_\infty\left(1-u\right)\right)\right]=1 \tag{68}$$

which yields after substituting $p_X t'_\infty\left(1-u\right)$ by its expression into (67) that:

$$\overline{n}'_X\left(u\right)=\frac{p_X n\left(X\right)\exp\left(-p_X t_\infty\left(1-u\right)\right)}{E_X\left[p_X\exp\left(-p_X t_\infty\left(1-u\right)\right)\right]} \tag{69}$$

$$=\frac{p_X\overline{n}_X\left(u\right)}{\sum_\ell p_{X^\ell}\overline{n}_{X^\ell}\left(u\right)} \tag{70}$$

and, as shown in Appendix A, this expression has a unique solution under the initial condition $\overline{n}\left(1,X\right)=n\left(X\right)$ which is $n\left(u\left|X\right.\right)$.

Step 4 can be established showing that the variance of $N\left(u^k\left|X\right.\right)/N$ tends to zero as N tends to infinity. Applying Lemma A1 with a_i a dummy for individual i having characteristics X, we get:

$$\begin{aligned} V\left[N\left(u^k\left|X\right.\right)/N\right] &= \sum_\ell N\left(X^\ell\right)\left[1_{\{X^\ell=X\}}-\frac{N(X)\,p_X^N\exp\left[-p_X^N t(N-k)/N\right]/N}{E_{emp}\left[p_X^N\exp\left[-p_X^N t(N-k)/N\right]\right]}\right]^2 \\ &\times\left[1-\exp\left[-p_{X^\ell}^N t(N-k)/N\right]\right]\exp\left[-p_{X^\ell}^N t(N-k)/N\right]/N^2 \\ &+r_V(N-k)/N^2 \end{aligned} \tag{71}$$

The term in brackets is bounded by one. As function $t(\cdot)$ is positive, $\exp\left[-p_X^N t(N-k)/N\right]$ is also bounded by one. This yields:

$$V\left[N\left(u^k\left|X\right.\right)/N\right]=O(1/N)+r_V(N-k)/N^2 \tag{72}$$

and this is enough to get: $V\left[N\left(u^k\left|X\right.\right)/N\right]\to 0$ as N tends to infinity.

2/ Case of a piece-wise constant function $\beta(.)$

We now consider the case where the coefficients of explanatory variables are piece-wise constant and we denote them as $\beta(u)$. These coefficients change value at knots υ^z such that $\varepsilon<\upsilon^1<...<\upsilon^Z=1$ with ε positive and small. We consider the asymptotic case where N, Z and N/Z (the average number of individuals by interval) all tend to infinity. We can apply the proof of case 1/ over the interval $\left(\upsilon^{Z-1},1\right]$ and we have:

$$p\lim E\left[N\left(u^k\left|X\right.\right)/N\right]=n\left(u\left|X\right.\right),\ \forall u\in(\upsilon_Z,1] \tag{73}$$

where:

$$n'\left(u\left|X\right.\right)=\frac{\exp\left[X\beta(1)\right]n\left(u\left|X\right.\right)}{\sum_\ell\exp\left[X^\ell\beta(1)\right]n\left(u\left|X^\ell\right.\right)} \tag{74}$$

with $n\left(1\left|X\right.\right)=n(X)$. This proves that the limit is $n\left(u\left|X\right.\right)$ as defined in the text on the interval $\left(\upsilon^Z,1\right]$. We are going to show recursively that this result holds at lower ranks. Assume that it is true down to rank υ^z and consider that $n\left(\upsilon^z\left|X\right.\right)=\liminf_{\upsilon^z<u}n\left(u\left|X\right.\right)$. Proof of case 1/ can be applied over the interval $\left(\upsilon^{z-1},\upsilon^z\right]$ and we have:

$$p\lim E\left[N\left(u^k\left|X\right.\right)/N\right]=n\left(u\left|X\right.\right),\ \forall u\in\left(\upsilon^{z-1},\upsilon^z\right] \tag{75}$$

where:

$$n'\left(u\left|X\right.\right)=\frac{\exp\left[X\beta(\upsilon^z)\right]n\left(u\left|X\right.\right)}{\sum_\ell\exp\left[X^\ell\beta(\upsilon^z)\right]n\left(u\left|X^\ell\right.\right)} \tag{76}$$

This proves that the limit is $n\left(u\left|X\right.\right)$ as defined in the text on the interval $\left(\upsilon^{z-1},\upsilon^z\right]$. After iterations, we get the result for the whole interval $[\varepsilon,1]$.

3/ Case of a continuous function $\beta(.)$

We now consider the case where the coefficients $\beta(u)$ are continuous and we use a step-wise approximation of $\beta(u)$ given by:

$$\beta_Z(u) = \sum_{z=1}^{Z} \beta\left(\frac{z}{Z}\right) 1\{z/Z - 1 < u \leq z/Z\} \tag{77}$$

We define by $n_Z(u|X)$ the measure of individuals with characteristics X still available for the job of rank u when the assignment mechanism is defined by $\beta_Z(u)$ instead of $\beta(u)$, and by $N_Z(u|X)$ the random variable that corresponds to the number of individuals with characteristics X still available at rank u under this assignment rule. The proportion of individuals in the population with characteristics X still available at rank u under this assignment rule is given by $P_Z(u|X) = N_Z(u|X)/N$. We have shown in case 2/ that $P_Z(u|X)$ tends to $n_Z(u|X)$ when N, Z and N/Z tend to infinity. The corresponding proportion when using the assignment rule defined by $\beta(u)$ is $P(u|X) = N(u|X)/N$, and we denote by $\Delta P(u|X) = P(u|X) - P_Z(u|X)$ the difference in the proportion of individuals under the two assignment rules.

We are going to show that for any $z \leq Z$ and rank $\upsilon^z = \lfloor N.z/Z \rfloor / N$, we have that:

$$\Delta P(\upsilon^z|X) = O\left(\frac{1}{Z^{1/2}}\right) + O\left(\frac{1}{N^{1/2}}\right) \tag{78}$$

uniformely over z in the sense that $O(\cdot)$ involves a bound that does not depend on z. The idea of the proof is to consider that:

$$\Delta P\left(\upsilon^{z-1}|X\right) = E_{\Omega(\upsilon^z)}\left[\Delta P\left(\upsilon^{z-1}|X\right)\right] + \varepsilon_{z,X} \tag{79}$$

where $\varepsilon_{z,X}$ is a random component depending on X and $E_{\Omega(\upsilon^z)}$ is the expectation operator conditional on the information available at rank υ^z, and then bound the right-hand side terms. Equation (78) will be enough together with case 2/ to get our main result. In more details, the proof can be decomposed into the following steps:

- 1. We first show that the random component verifies:

$$\varepsilon_{z,X} = O\left(Z^{-1/2}\right) + O\left(N^{-1/2}\right) \tag{80}$$

  2. We will then show that we have:

$$E_{\Omega(\upsilon^z)}\left[\Delta P\left(\upsilon^{z-1}|X\right)\right] = \Delta P\left(\Delta\upsilon^z|X\right) + O\left(\frac{1}{Z^2}\right) + O\left(\frac{1}{Z}\right)\max_{\ell}\left[\Delta P\left(\upsilon^z\left|X^{\ell}\right.\right)\right] \tag{81}$$

  We then obtain (78) by recurrence. Consider first the case $z = Z$. The relationship (78) holds trivially since $N_Z\left(\upsilon^Z|X\right) = N\left(\upsilon^Z|X\right) = N(X)$ as $\upsilon^Z = 1$. Then, supposing that (78) is true for a given z, it is straightfoward from (79), (80) and (81) to show that it is also true for $z - 1$.

3. Finally, the relationship (78) is enough to show that $E\left[N\left(u^{\lfloor vN\rfloor+1}\left|X\right.\right)\right]/N \underset{a.s.}{\rightarrow} n\left(v\left|X\right.\right)$ when N tends to infinity and Z is chosen such that Z and N/Z tend to infinity (this is for instance the case when $Z=N^{1/2}$). This can be done by applying the results of case 2/ to the piece-wise constant function $\beta_Z\left(.\right)$ since there is then an infinity of positions in each of the intervals defined by the nodes.

Step 1 can be completed considering that the processes underlying $P\left(\upsilon^{z-1}\left|X\right.\right)$ et $P_Z\left(\upsilon^{z-1}\left|X\right.\right)$ are independent, as we have:

$$V_{\Omega(\upsilon^z)}\left(\varepsilon_{z,X}\right)=V_{\Omega(\upsilon^z)}\left[P\left(\upsilon^z\left|X\right.\right)-P\left(\upsilon^{z-1}\left|X\right.\right)\right]+V_{\Omega(\upsilon^z)}\left[P_Z\left(\upsilon^z\left|X\right.\right)-P_Z\left(\upsilon^{z-1}\left|X\right.\right)\right] \tag{82}$$

where $V_{\Omega(\upsilon^z)}$ is the variance operator conditional on the information available at rank υ^z. By construction,

$$\begin{aligned} 0 &\leq P\left(\upsilon^z\left|X\right.\right)-P\left(\upsilon^{z-1}\left|X\right.\right) \\ &\leq \frac{\left\lfloor N\frac{z}{Z}\right\rfloor-\left\lfloor N\frac{z-1}{Z}\right\rfloor}{N} && (83) \\ &\leq \frac{1}{Z}+\frac{1}{N}=O\left(\frac{1}{Z}\right)+O\left(\frac{1}{N}\right) && (84) \end{aligned}$$

The same inequality holds for $P_Z\left(\upsilon^z\left|X\right.\right)-P_Z\left(\upsilon^{z-1}\left|X\right.\right)$ and it yields:

$$V_{\Omega(\upsilon^z)}\left(\varepsilon_{z,X}\right)=O\left(\frac{1}{Z}\right)+O\left(\frac{1}{N}\right) \tag{85}$$

which leads to equation (80).

Step 2 can be shown considering that we have:

$$E_{\Omega(\upsilon^z)}\left[P\left(\upsilon^z\left|X\right.\right)-P\left(\upsilon^{z-1}\left|X\right.\right)\right]=\frac{1}{N}\sum_{k=\left\lfloor N\frac{z-1}{Z}\right\rfloor+1}^{\left\lfloor N\frac{z}{Z}\right\rfloor}E_{\Omega(\upsilon^z)}\left[\frac{P\left(u^k\left|X\right.\right)\exp\left(X\beta\left(u^k\right)\right)}{\sum_{\ell}P\left(u^k\left|X^{\ell}\right.\right)\exp\left(X^{\ell}\beta\left(u^k\right)\right)}\right] \tag{86}$$

To ease the notations in the sequel, we denote the length of an interval as:

$$\Delta_z=\left(\left\lfloor N\frac{z}{Z}\right\rfloor-\left\lfloor N\frac{z-1}{Z}\right\rfloor\right)/N \tag{87}$$

where N and Z are not introduced as subscripts to ease the readability. For all values of X and all values of k such that $\left\lfloor N\frac{z-1}{Z}\right\rfloor+1\leq k\leq\left\lfloor N\frac{z}{Z}\right\rfloor$, we have:

$$P\left(\upsilon^{z-1}\left|X\right.\right)\leq P\left(u^k\left|X\right.\right)\leq P\left(\upsilon^z\left|X\right.\right) \tag{88}$$

A lower bound for $P\left(\upsilon^{z-1}\left|X\right.\right)$ is the decrease of the proportion $P\left(\upsilon^{z}\left|X\right.\right)$ when all the jobs with ranks between υ^{z-1} and υ^{z} are filled by individuals with characteristics X, and thus we have the following inequalities:

$$\begin{aligned}P\left(\upsilon^{z}\left|X\right.\right)-\Delta_{z} &\leq P\left(u^{k}\left|X\right.\right)\leq P\left(\upsilon^{z}\left|X\right.\right) &(89)\\ \min_{\frac{z-1}{Z}\leq k\leq\frac{z}{Z}}\exp\left(X\beta\left(u^{k}\right)\right) &\leq \exp\left(X\beta\left(u^{k}\right)\right)\leq\max_{\frac{z-1}{Z}\leq k\leq\frac{z}{Z}}\exp\left(X\beta\left(u^{k}\right)\right) &(90)\end{aligned}$$

Using these inequalities, we obtain that:

$$\begin{aligned}&\Delta_{z}\frac{\left(P\left(\upsilon^{z}\left|X\right.\right)-\Delta_{z}\right)\min\limits_{\frac{z-1}{Z}\leq k\leq\frac{z}{Z}}\exp\left(X\beta\left(u^{k}\right)\right)}{\sum\limits_{\ell}P\left(\upsilon^{z}\left|X^{\ell}\right.\right)\max\limits_{\frac{z-1}{Z}\leq k\leq\frac{z}{Z}}\exp\left(X^{\ell}\beta\left(u^{k}\right)\right)}\\ \leq\ &\frac{1}{N}\sum_{k=\left\lfloor N\frac{z-1}{Z}\right\rfloor+1}^{\left\lfloor N\frac{z}{Z}\right\rfloor}E_{\Omega(\upsilon^{z})}\left[\frac{P\left(u^{k}\left|X\right.\right)\exp\left(X\beta\left(u^{k}\right)\right)}{\sum\limits_{\ell}P\left(u^{k}\left|X^{\ell}\right.\right)\exp\left(X^{\ell}\beta\left(u^{k}\right)\right)}\right]\leq &(91)\\ &\Delta_{z}\frac{P\left(\upsilon^{z}\left|X\right.\right)\max\limits_{\frac{z-1}{Z}\leq k\leq\frac{z}{Z}}\exp\left(X\beta\left(u^{k}\right)\right)}{\sum\limits_{\ell}\left[P\left(\upsilon^{z}\left|X^{\ell}\right.\right)-\Delta_{z}\right]\min\limits_{\frac{z-1}{Z}\leq k\leq\frac{z}{Z}}\exp\left(X^{\ell}\beta\left(u^{k}\right)\right)} &(92)\end{aligned}$$

and from (86), we get:

$$\begin{aligned}&\frac{\left(P\left(\upsilon^{z}\left|X\right.\right)-\Delta_{z}\right)\min\limits_{\frac{z-1}{Z}\leq k\leq\frac{z}{Z}}\exp\left(X\beta\left(u^{k}\right)\right)}{\sum\limits_{\ell}P\left(\upsilon^{z}\left|X^{\ell}\right.\right)\max\limits_{\frac{z-1}{Z}\leq k\leq\frac{z}{Z}}\exp\left(X^{\ell}\beta\left(u^{k}\right)\right)}\\ \leq\ &\frac{E_{\Omega(\upsilon^{z})}\left[P\left(\upsilon^{z}\left|X\right.\right)-P\left(\upsilon^{z-1}\left|X\right.\right)\right]}{\Delta_{z}}\leq &(93)\\ &\frac{P\left(\upsilon^{z}\left|X\right.\right)\max\limits_{\frac{z-1}{Z}\leq k\leq\frac{z}{Z}}\exp\left(X\beta\left(u^{k}\right)\right)}{\sum\limits_{\ell}\left[P\left(\upsilon^{z}\left|X^{\ell}\right.\right)-\Delta_{z}\right]\min\limits_{\frac{z-1}{Z}\leq k\leq\frac{z}{Z}}\exp\left(X^{\ell}\beta\left(u^{k}\right)\right)} &(94)\end{aligned}$$

We can get another inequality using the fact that $\beta_{Z}\left(u\right)$ converges uniformly to $\beta\left(u\right)$, as there exists a constant denoted C such that for all X and k such that $\left\lfloor N\frac{z-1}{Z}\right\rfloor+1\leq k\leq\left\lfloor N\frac{z}{Z}\right\rfloor$, we have:

$$\left|\exp X\beta\left(u^{k}\right)-\exp X\beta_{Z}\left(\frac{z}{Z}\right)\right|<\frac{C}{Z} \tag{95}$$

Using this inequality and the fact that $\beta_Z\left(\frac{z}{Z}\right)=\beta\left(\frac{z}{Z}\right)$, we deduce from (94) that:

$$\begin{aligned}&\frac{\left(P\left(\upsilon^z\left|X\right.\right)-\Delta_z\right)\left[\exp\left(\beta\left(\frac{z}{Z}\right)\right)-\frac{C}{Z}\right]}{\sum\limits_\ell P\left(\upsilon^z\left|X^\ell\right.\right)\left[\exp\left(\beta\left(\frac{z}{Z}\right)\right)+\frac{C}{Z}\right]}\\ \leq\;&\frac{E_{\Omega(\upsilon^z)}\left[P\left(\upsilon^z\left|X\right.\right)-P\left(\upsilon^{z-1}\left|X\right.\right)\right]}{\Delta_z}\leq \qquad (96)\\ &\frac{P\left(\upsilon^z\left|X\right.\right)\left[\exp\left(\beta\left(\frac{z}{Z}\right)\right)+\frac{C}{Z}\right]}{\sum\limits_\ell\left[P\left(\upsilon^z\left|X^\ell\right.\right)-\Delta_z\right]\left[\exp\left(\beta\left(\frac{z}{Z}\right)\right)-\frac{C}{Z}\right]} \qquad (97)\end{aligned}$$

An inequality similar to (94) holds when considering the assignment rule given by $\beta^Z(u)$, and we have:

$$\begin{aligned}&\frac{\left[P_Z\left(\upsilon^z\left|X\right.\right)-\Delta_z\right]\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum\limits_\ell P_Z\left(u^k\left|X^\ell\right.\right)\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)}\\ \leq\;&\frac{E_{\Omega(\upsilon^z)}\left[P\left(\upsilon^z\left|X\right.\right)-P\left(\upsilon^{z-1}\left|X\right.\right)\right]}{\Delta_z}\leq \qquad (98)\\ &\frac{P_Z\left(\upsilon^z\left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum\limits_\ell\left[P_Z\left(\upsilon^z\left|X^\ell\right.\right)-\Delta_z\right]\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)} \qquad (99)\end{aligned}$$

Computing the difference between (97) and (99), we get the inequality:

$$\begin{aligned}&E_{\Omega(\upsilon^z)}\left[\frac{\Delta P\left(\upsilon^z\left|X\right.\right)}{\Delta_z}\right]-E_{\Omega(\upsilon^z)}\left[\frac{\Delta P\left(\upsilon^{z-1}\left|X\right.\right)}{\Delta_z}\right]\\ \geq\;&\frac{\left(P\left(\upsilon^z\left|X\right.\right)-\Delta_z\right)\left[\exp\left(X\beta\left(\frac{z}{Z}\right)\right)-\frac{C}{Z}\right]}{\sum\limits_\ell P\left(\upsilon^z\left|X^\ell\right.\right)\left[\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)+\frac{C}{Z}\right]}-\frac{P_Z\left(\upsilon^z\left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum\limits_\ell\left[P_Z\left(\upsilon^z\left|X^\ell\right.\right)-\Delta_z\right]\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)} \qquad (100)\end{aligned}$$

We are going to make Taylor expansions of the right-hand side terms of (100). For the denominator of the first term, we have:

$$\begin{aligned}\sum_\ell P\left(\upsilon^z\left|X^\ell\right.\right)\left[\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)+\frac{C}{Z}\right]&=\sum_\ell P_Z\left(\upsilon^z\left|X^\ell\right.\right)\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)+\frac{C}{Z}\sum_\ell P_Z\left(\upsilon^z\left|X^\ell\right.\right)\\ &+\sum_\ell\Delta P\left(\upsilon^z\left|X^\ell\right.\right)\left[\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)+\frac{C}{Z}\right] \qquad (101)\end{aligned}$$

Denoting $\Psi_z=\max_\ell\left[\Delta P\left(\upsilon^z\left|X^\ell\right.\right)\right]$, we get from (101) that:

$$\sum_\ell P\left(\upsilon^z\left|X^\ell\right.\right)\left[\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)+\frac{C}{Z}\right]=\sum_\ell P_Z\left(\upsilon^z\left|X^\ell\right.\right)\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)+O\left(\frac{1}{Z}\right)+O\left(\Psi_z\right) \qquad (102)$$

As we also have $\Delta_z = O\left(Z^{-1}\right)$, the numerator of the first right-hand side term of (100) can be rewritten such that:

$$\left(P\left(\upsilon^z\left|X\right.\right)-\Delta_z\right)\left[\exp\left(X\beta\left(\frac{z}{Z}\right)\right)-\frac{C}{Z}\right]=P\left(\upsilon^z\left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)+O\left(\frac{1}{Z}\right) \tag{103}$$

Using (102) and (103), and making an additional Taylor expansion, we get that:

$$\begin{aligned}
&\frac{\left(P\left(\upsilon^z\left|X\right.\right)-\Delta_z\right)\left[\exp\left(X\beta\left(\frac{z}{Z}\right)\right)-\frac{C}{Z}\right]}{\sum_{\ell}P\left(\upsilon^z\left|X^{\ell}\right.\right)\left[\exp\left(X^{\ell}\beta\left(\frac{z}{Z}\right)\right)+\frac{C}{Z}\right]}\\
=\;&\frac{P\left(\upsilon^z\left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)+O\left(\frac{1}{Z}\right)}{\sum_{\ell}P\left(\upsilon^z\left|X^{\ell}\right.\right)\exp\left(X^{\ell}\beta\left(\frac{z}{Z}\right)\right)}\left[1+O\left(\frac{1}{Z}\right)+O\left(\Psi_z\right)\right] &(104)\\
=\;&\frac{P\left(\upsilon^z\left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum_{\ell}P_Z\left(\upsilon^z\left|X^{\ell}\right.\right)\exp\left(X^{\ell}\beta\left(\frac{z}{Z}\right)\right)}+O\left(\frac{1}{Z}\right)+O\left(\Psi_z\right) &(105)
\end{aligned}$$

Similarly, we can rewrite the second right-hand side term of (100) as:

$$\begin{aligned}
\frac{P_Z\left(\upsilon^z\left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum_{\ell}\left[P_Z\left(\upsilon^z\left|X^{\ell}\right.\right)-\Delta_z\right]\exp\left(X^{\ell}\beta\left(\frac{z}{Z}\right)\right)} &= \frac{P_Z\left(\upsilon^z\left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum_{\ell}P_Z\left(\upsilon^z\left|X^{\ell}\right.\right)\exp\left(X^{\ell}\beta\left(\frac{z}{Z}\right)\right)}\left[1+O\left(\frac{1}{Z}\right)\right] &(106)\\
&= \frac{P_Z\left(\upsilon^z\left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum_{\ell}P_Z\left(\upsilon^z\left|X^{\ell}\right.\right)\exp\left(X^{\ell}\beta\left(\frac{z}{Z}\right)\right)}+O\left(\frac{1}{Z}\right) &(107)
\end{aligned}$$

This yields after an additional Taylor expansion that:

$$\begin{aligned}
&E_{\Omega(\upsilon^z)}\left[\frac{\Delta P\left(\upsilon^z\left|X\right.\right)}{\Delta_z}\right]-E_{\Omega(\upsilon^z)}\left[\frac{\Delta P\left(\upsilon^{z-1}\left|X\right.\right)}{\Delta_z}\right]\\
\geq\;&\frac{\Delta P\left(\upsilon^z\left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum_{\ell}P_Z\left(\upsilon^z\left|X^{\ell}\right.\right)\exp\left(X^{\ell}\beta\left(\frac{z}{Z}\right)\right)}+O\left(\frac{1}{Z}\right)+O\left(\Psi_z\right) &(108)\\
\geq\;&O\left(\frac{1}{Z}\right)+O\left(\Psi_z\right) &(109)
\end{aligned}$$

In a similar way, we have:

$$\begin{aligned}
&E_{\Omega(\upsilon^z)}\left[\frac{\Delta P\left(\upsilon^z\left|X\right.\right)}{\Delta_z}\right]-E_{\Omega(\upsilon^z)}\left[\frac{\Delta P\left(\upsilon^{z-1}\left|X\right.\right)}{\Delta_z}\right]\\
\leq\;&\frac{P\left(\upsilon^z\left|X\right.\right)\left[\exp\left(X\beta\left(\frac{z}{Z}\right)\right)+\frac{C}{Z}\right]}{\sum_{\ell}\left[P\left(\upsilon^z\left|X^{\ell}\right.\right)-\Delta_z\right]\left[\exp\left(X^{\ell}\beta\left(\frac{z}{Z}\right)\right)-\frac{C}{Z}\right]}-\frac{\left[P_Z\left(\upsilon^z\left|X\right.\right)-\Delta_z\right]\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum_{\ell}P_Z\left(u^k\left|X^{\ell}\right.\right)\exp\left(X^{\ell}\beta\left(\frac{z}{Z}\right)\right)} &(110)
\end{aligned}$$

Using the same line of arguments as before, we get:

$$\begin{aligned}\sum_\ell \left[P\left(\upsilon^z \left|X^\ell\right.\right)-\Delta_z\right]\left[\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)-\frac{C}{Z}\right] &= \sum_\ell P_Z\left(\upsilon^z \left|X^\ell\right.\right)\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)+O\left(\frac{1}{Z}\right)\\ &\quad +O\left(\Psi_z\right) \qquad (111)\\ P\left(\upsilon^z \left|X\right.\right)\left[\exp\left(X\beta\left(\frac{z}{Z}\right)\right)+\frac{C}{Z}\right] &= P\left(\upsilon^z \left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)+O\left(\frac{1}{Z}\right) \qquad (112)\\ \left[P_Z\left(\upsilon^z \left|X\right.\right)-\Delta_z\right]\exp\left(X\beta\left(\frac{z}{Z}\right)\right) &= P_Z\left(\upsilon^z \left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)+O\left(\frac{1}{Z}\right) \qquad (113)\end{aligned}$$

Hence:

$$\begin{aligned}&\frac{P\left(\upsilon^z \left|X\right.\right)\left[\exp\left(X\beta\left(\frac{z}{Z}\right)\right)+\frac{C}{Z}\right]}{\sum_\ell \left[P\left(\upsilon^z \left|X^\ell\right.\right)-\Delta_z\right]\left[\exp\left(X\beta\left(\frac{z}{Z}\right)\right)-\frac{C}{Z}\right]}\\ =\ &\frac{P\left(\upsilon^z \left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)+O\left(\frac{1}{Z}\right)}{\sum_\ell P_Z\left(\upsilon^z \left|X^\ell\right.\right)\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)}\left[1+O\left(\frac{1}{Z}\right)+O\left(\Psi_z\right)\right] \qquad (114)\\ =\ &\frac{P\left(\upsilon^z \left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum_\ell P_Z\left(\upsilon^z \left|X^\ell\right.\right)\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)}+O\left(\frac{1}{Z}\right)+O\left(\Psi_z\right) \qquad (115)\end{aligned}$$

Similarly:

$$\begin{aligned}\frac{\left[P_Z\left(\upsilon^z \left|X\right.\right)-\Delta_z\right]\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum_\ell P_Z\left(u^k \left|X^\ell\right.\right)\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)} &= \frac{P_Z\left(\upsilon^z \left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)+O\left(\frac{1}{Z}\right)}{\sum_\ell P_Z\left(u^k \left|X^\ell\right.\right)\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)} \qquad (116)\\ &= \frac{P_Z\left(\upsilon^z \left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum_\ell P_Z\left(\upsilon^z \left|X^\ell\right.\right)\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)}+O\left(\frac{1}{Z}\right) \qquad (117)\end{aligned}$$

Finally, we obtain:

$$\begin{aligned}&E_{\Omega(\upsilon^z)}\left[\frac{\Delta P\left(\upsilon^z \left|X\right.\right)}{\Delta_z}\right]-E_{\Omega(\upsilon^z)}\left[\frac{\Delta P\left(\upsilon^{z-1} \left|X\right.\right)}{\Delta_z}\right]\\ \leq\ &\frac{\Delta P\left(\upsilon^z \left|X\right.\right)\exp\left(X\beta\left(\frac{z}{Z}\right)\right)}{\sum_\ell P_Z\left(\upsilon^z \left|X^\ell\right.\right)\exp\left(X^\ell\beta\left(\frac{z}{Z}\right)\right)}+O\left(\frac{1}{Z}\right)+O\left(\Psi_z\right) \qquad (118)\\ \leq\ &O\left(\frac{1}{Z}\right)+O\left(\Psi_z\right) \qquad (119)\end{aligned}$$

From (109) and (119), we get a lower bound and an upper bound for our quantity of interest that are associated to the same convergence speed. As a consequence, we obtain:

$$E_{\Omega(\upsilon^z)}\left[\frac{\Delta P\left(\upsilon^z \left|X\right.\right)}{\Delta_z}\right]-E_{\Omega(\upsilon^z)}\left[\frac{\Delta P\left(\upsilon^{z-1} \left|X\right.\right)}{\Delta_z}\right]=O\left(\frac{1}{Z}\right)+O\left(\Psi_z\right) \qquad (120)$$

which can be rewritten as:

$$E_{\Omega(v^z)}\left[\Delta P\left(v^{z-1}\left|X\right.\right)\right] = E_{\Omega(v^z)}\left[\Delta P\left(v^z\left|X\right.\right)\right] + O\left(\frac{\Delta_z}{Z}\right) + O\left(\Delta_z\Psi_z\right) \tag{121}$$

Using the fact that $E_{\Omega(v^z)}\left[\frac{\Delta P(v^z|X)}{\Delta_z}\right] = \frac{\Delta P(v^z|X)}{\Delta_z}$ and $\Delta_z = O\left(Z^{-1}\right)$, we finally obtain expression (81).

Step 3 can then be completed to obtain our main result. First note that, for any $v \in (0,1]$, we have that $v^{z^N-1} \leq \lfloor vN \rfloor / (N-1) \leq v^{z^N}$ for some z^N and this yields:

$$P\left(v^{z^N-1}\left|X\right.\right) \leq N\left(u^{\lfloor vN \rfloor}\left|X\right.\right)/N \leq P\left(v^{z^N}\left|X\right.\right) \tag{122}$$

It is thus enough to show that both $P\left(v^{z^N-1}\left|X\right.\right)$ and $P\left(v^{z^N}\left|X\right.\right)$ tend to $n\left(v\left|X\right.\right)$ to prove our result. We have the following decomposition:

$$\begin{aligned} P\left(v^{z^N}\left|X\right.\right) - n\left(v\left|X\right.\right) &= \left[P\left(v^{z^N}\left|X\right.\right) - P_Z\left(v^{z^N}\left|X\right.\right)\right] + \left[P_Z\left(v^{z^N}\left|X\right.\right) - n_Z\left(v^{z^N}\left|X\right.\right)\right] \\ &\quad + \left[n_Z\left(v^{z^N}\left|X\right.\right) - n\left(v\left|X\right.\right)\right] \end{aligned} \tag{123}$$

where z^N is such that $v^{z^N-1} \leq \lfloor vN \rfloor / (N-1) \leq v^{z^N}$.

It can be shown that the first right hand-side term converges to zero as N and Z tend to infinity using result (78). We have already shown that the second right-hand side term converges to zero provided that the number of individuals in each interval, which is approximately N/Z tends to infinity. Finally, the third right-hand side term converges to zero between v^{z^N} converges to v and $\beta_Z(u)$ converges to $\beta(u)$ uniformely over the $[0,1]$ interval. Indeed, $n_Z\left(v\left|X\right.\right)$ and $n\left(v\left|X\right.\right)$ verify the same differential equation at infinity, which is given by (8), and this equation has a unique solution for the initial conditions $n(X)$. This yields that $P\left(v^{z^N}\left|X\right.\right)$ converges to $n\left(v\left|X\right.\right)$. A similar proof can be applied to show that $P\left(v^{z^N-1}\left|X\right.\right)$ converges to $n\left(v\left|X\right.\right)$, and we finally get that $N\left(u^{\lfloor vN \rfloor}\left|X\right.\right)/N$ converges to $n\left(v\left|X\right.\right)$.